A very bitter chocolate problem.

A new kind of Nim.

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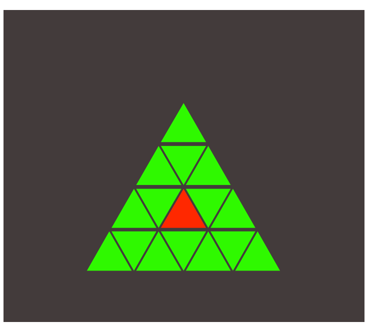
Introduction. Here we have a very interesting game. We are going to introduce a new type of Nim. By this problem we are going to introduce a new field in the research of Nim! This research has been already introduced at MathPuzzle.com.

Problem 1 Suppose that you have the following chocolate. See Graph (1). The green parts are sweet, but the red part is very bitter. Two players in turn breaks the chocolate (in a straight line along the grooves) and eats the piece he breaks off. The player to leave his opponent with the single bitter part is the winner. Describe the stratedy to ensure winning.



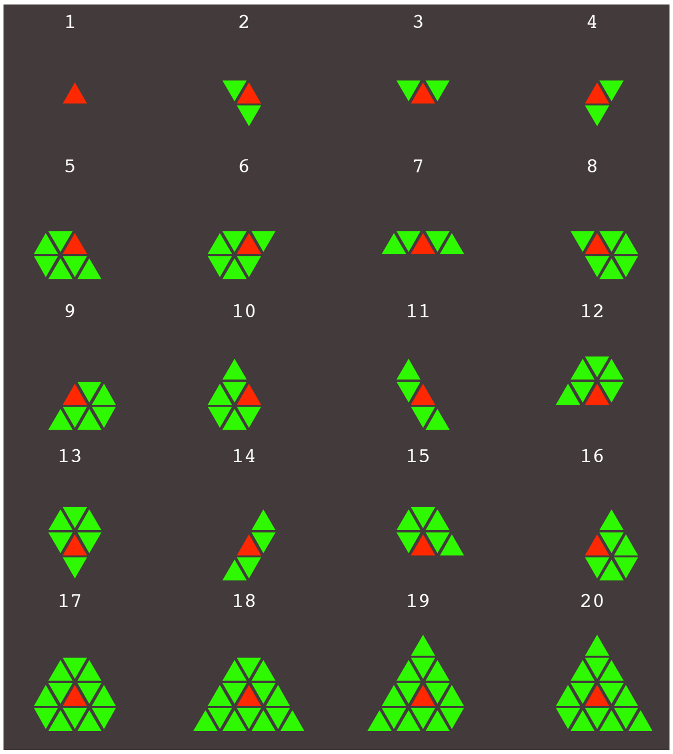
……Graph (1)

Solution. In this game there are two kinds of positions. One kind is a **P-position**, a previous-player-winning position. The other is an **N-position**, a Next-player-winning position. Let me explain about these positions. In the followings we use the word **option** to mean "choice of move".

P - positions	**N - Positions**
Every **option** leads to an N - position	There is always at least one **option** leading to a **P - position**

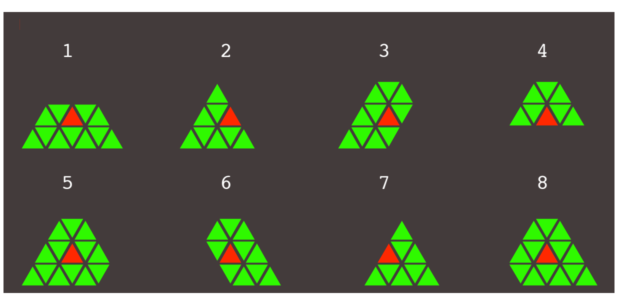
By using computer we can find all the **P-positions.** (For this scale of problem you can find all the **P-position** only with pen and paper.) This Graph(2) contains all the **P-positions.** All the other positions are **N-position.** As you can see easily the original position in Graph(1) is an **N-position,** because you cannot find the original position in Graph(2).

Therefore you are sure to win if you start the game as the first player. The strategy is easy. Let me explain how to use the strategy by an example.



……Graph (2)

An example of a stratedy. From the original position you can move to the 18th, 19th or 20th positions in Graph (2). Suppose that you chose the 18th position. Then your opponent will choose one of the following 8 positions.

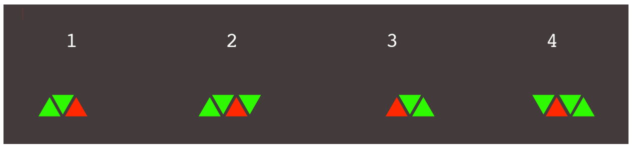


……Graph (3)

If the opponent chose the first option of Graph(3), then you can choose the 7th of Graph(2). Then your opponent choose one of the following 4 positions.
If the opponent choose the 1th or the 3rd position, you can take away the green part and win the game.

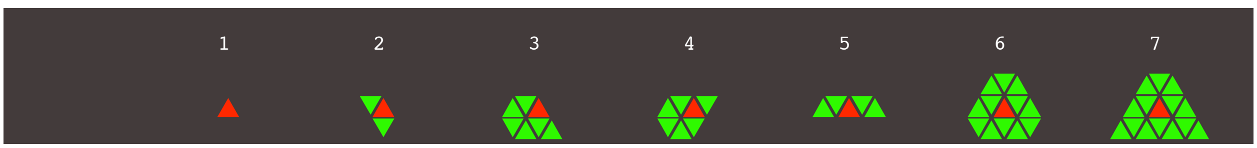
If the opponent choose the 2nd or 4th position of Graph (4), you can move to the 3rd of Graph(2). After that the opponent will take the right or left green part, then you can take the remainning green part and win.

The stratedy is clear. If you start with the original position in Graph(1), then you have only to move to a **P-position**. By the next move your opponent will move to **N-position**. From this position you have only to a **P-position**. By continuing this process you are sure to win.



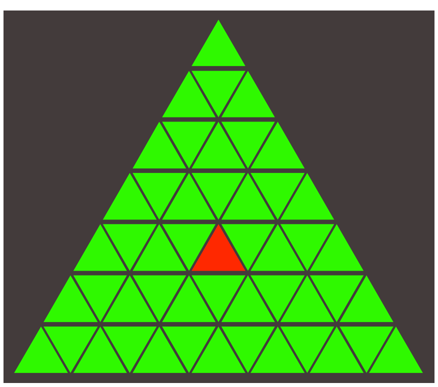
……Graph (4)

Remark. If you are going to play this game and want to memorize all the **P-positions**, it is better to make the list of **P-positions** smaller. In Graph(2) the 2nd, 3rd and 4th positions are essentially the same, and the 6th and 8th positions are essentially the same, too. By considering this we can choose 7 positions out of 20 positions in Graph(2), and we can make the following Graph(5).



……Graph (5)

Problem 2 Suppose that you have the following chocolate. See Graph (6). The rule is the same as in Problem 1. Describe the stratedy to ensure winning.

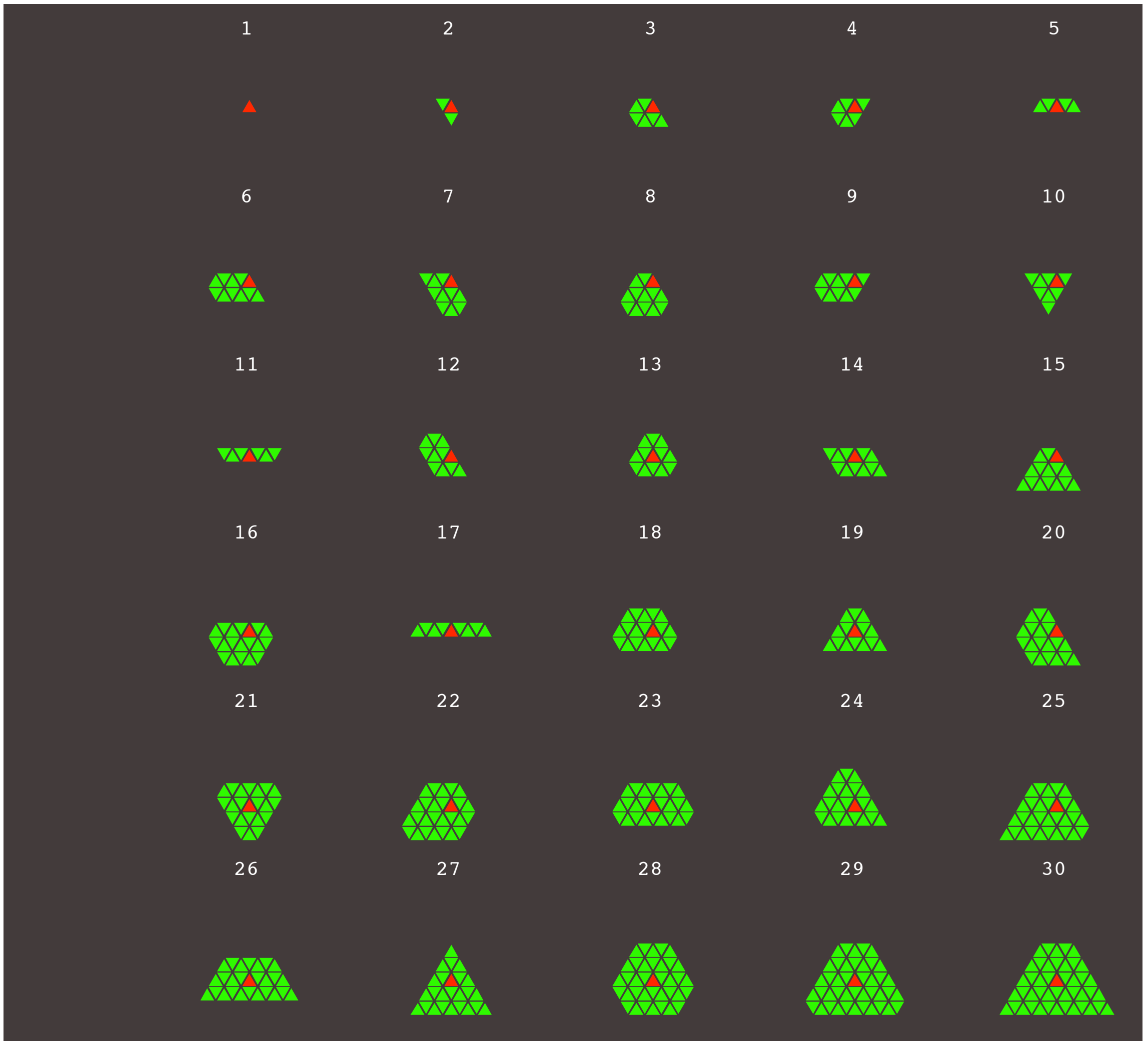


……Graph (6)

By using computer we can find all the **P-positions.** (This time it will be very hard to find all the **P-position** only with pen and paper.) We have 118 **P-positions.** Instead of displaying all the **P-positions** we are going to choose a part of them. We did the same thing when we made Graph (5). This time we choose and present 30 of all **P-positions.** The chosen **P-positions** are presented in Graph (7).

As you can see easily, the original position in Graph (6) is an **N-position**, because you cannot find the original position in Graph (7).

Therefore you are sure to win if you start the game as the first player. The strategy is easy and you can use the same kind of stratedy you used in the previous problem.



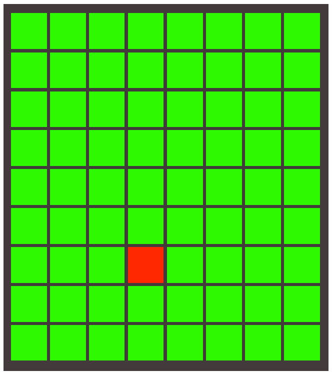
……Graph (7)

Mathematical background. The theory of Nim is very rich in its contents. There are also many disguises for the game of Nim. For example the poisoned chocolate problem is one of them. Robin [4] introduced this problem in the Mathematical Gazette. We are going to explain about his problem and compare it to our problem. The solution of this game is the same as that of the traditional Nim, and the solution of traditional Nim was given by C.Bouton [1].

We are going to solve the poisoned chocolate problem that was by A.C.Robin [4]. We are going to change the numbers of parts.

Example 1. This time you have the following chocolate. See Graph(8). The rules are the same as in problme 1 and 2. Describe the stratedy to ensure winning.

Solution. We are going to use the method made by C.Bouton [1]. This chocolate has 6 rows over the red part, 4 columns on the right side, 2 rows under the red parts and 3 columns on the left side of the red part. We partitioned these numbers 6,4,2,3 into powers of two. Therefore 6 = 2 + 4, 4 = 4, 2 = 2 and 3 = 1 + 2. Thus we can make Table(1).



……Graph (8)

orientation	over	right	under	left
numbers	6	4	2	3
1	0	0	0	1
2	1	0	1	1
4	1	1	0	0

……Table (1)

We are going to check the table for each power of 2. For $2^0 = 1$ we have only 1 occurrence. For $2^1 = 2$ we have 3 occurrences, and for $2^2 = 4$ we have 2 occurrences. C.Bouton [1] proved that we have a **P-position** when each power of 2 occurs evenly often. Therefore we have an **N-position** in Graph (8). We have to remove one row over the red part if you want to move to a **P-position**. See Table (2). Here each power of 2 occurs evenly often.

orientation	over	right	under	left
numbers	5	4	2	3
1	1	0	0	1
2	0	0	1	1
4	1	1	0	0

……Table (2)

Another option is to remove 3 columns from the left side of the red part.

Theorem. In the game of above chocolate problem we have a **P-position** when each power of 2 occurs evenly often.
We omit the proof, because this is a well known fact about the game of Nim.

Now we are going to compare Problem 1,2 and Example 1. Example 1 is a typical example of traditional Nim. We can represent the game with 4-numbers $\{x_1, x_2, x_3, x_4\}$. When it is your turn to move, you choose one of the 4 coodinates and take a number that is smaller than the coodinate from it. These 4 coodinates are independent, i.e., you can take a number from one coodinate without affecting other coodinate.

Problem 1 and 2 are different. You can cut the chocolate in 6 ways, so it is appropriate to represent it with 6 numbers $\{x_1, x_2, x_3, x_4, x_5, x_6\}$. We represent the position in Graph(1) with {2,1,2,1,2,1} and the 6th position in Graph(2) with {0,1,0,1,1,1}.

In Problem 1 and 2 these 6 coodinates are not independent. We have 6 inequalities between these 6 coodinates.

$x_1 \leq x_2 + x_6$, $x_2 \leq x_1 + x_3 + 1$, $x_3 \leq x_2 + x_4$,

$x_4 \leq x_3 + x_5 + 1$, $x_5 \leq x_4 + x_6$, $x_6 \leq x_5 + x_1$

As far as we know there have been no research on the Nim conditioned by inequalities. Therefore we are studying a very new kind of Nim. In this article we only used computer programming to find a winning stratedy, because we could find a mathematical solutions for our problems.

We have been studying more simple type of Nim conditioned by inequalities, and have managed to find a mathematical solutions. We talked about it at Gama Amusement Society 2005 in Japan, and now writing a paper for a math journal.

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